Permutation and Rearrangement[1]

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0. Purpose

We are interested in the use of a group of bijections of a set of *n* elements to describe the rigid motions of a *n*-sided regular polygon. Following Hunter (1966), we will call these bijections "permutations" or "rearrangements", depending on whether they stand for the transformation of a figure or for the transformation of its embedding space. In this paper, we construct a group from the set of all bijective mappings of three elements, and define rearrangement as an isomorphism on this group. In section 3, we develop a three-point geometry, defined as a group action on a coset space. Finally, we briefly explore the implications of our construction for geometric representation generally.

1. Permutation

The use of permutations in a group structure to describe rigid geometric transformations (which preserve properties like size, distance, angles, etc.) presupposes that permutations be defined as bijective functions – that is, functions that preserve properties of a set they act on, such as the number of elements in the set or the number of times each element appears in a permuted ordering. Bijections preserve such properties by definition because they map a domain to a co-domain such that each element of the domain maps to exactly one element of the codomain; and each element of the codomain has a map to it from the domain. Specifically, permutations are *self-maps*, as their domain and codomain are identical.

For a geometric construction, a reading of these self-maps can be double, in the sense that permuted elements can stand for the vertices of a figure or for the spatial positions such figures occupy. Hunter (1966) calls this latter case "rearrangement", as opposed to "permutation" (which stands for transformations of the figure itself).[2] These two different expressions correspond to one another as far as they both describe the same transformation, albeit from two perspectives: one for the transformation of a figure with reference to fixed positions; and another for space-transformations of positions with reference to a fixed figure.[3]

Below, we define permutation as a bijective function and give an example involving the set of all bijections of three elements.

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[1] This paper is dedicated in memory of Jean-Michel Vappereau (1948-2025). I am indebted to others who worked with me on this paper, especially R. Groome and A. Lotfalian of PLACE (Psychoanalysis Los Angeles Extension) during the cartel and seminar semester of 2025.

[2] Hunter, D.B. "Permutations and Rearrangements" *Mathematical Gazette*, 1966, p. 290. Note the term "permutation" is used there in both a general and particular sense, as Hunter calls rearrangement a "permutation of the positions", whereas permutation is a permutation of "objects", meaning geometric figures.

[3] C.f. Budden, F.J. *Fascination of Groups*, pgs. 200-201.

Definition 1.0. (Permutation): We define a *permutation* as a bijective self-map of a set **A** of differential[4] elements, denoted **f:A→A**.

Example 1.0. Let **X** denote a set of elements **{0,1,2}** and **Bij(X)** the set of all bijective mappings of these elements. **Bij(X)** has 3!=6 such maps, divided into three types, based on how many elements in a particular bijection remain *fixed* (i.e., mapped to themselves). A *transposition* is a map replacing two elements and fixing a third; a *cycle* replaces all three elements, leaving none fixed; and an *identity* fixes all three elements. **Bij(X)** has three transpositions[5], two cycles, and one identity.

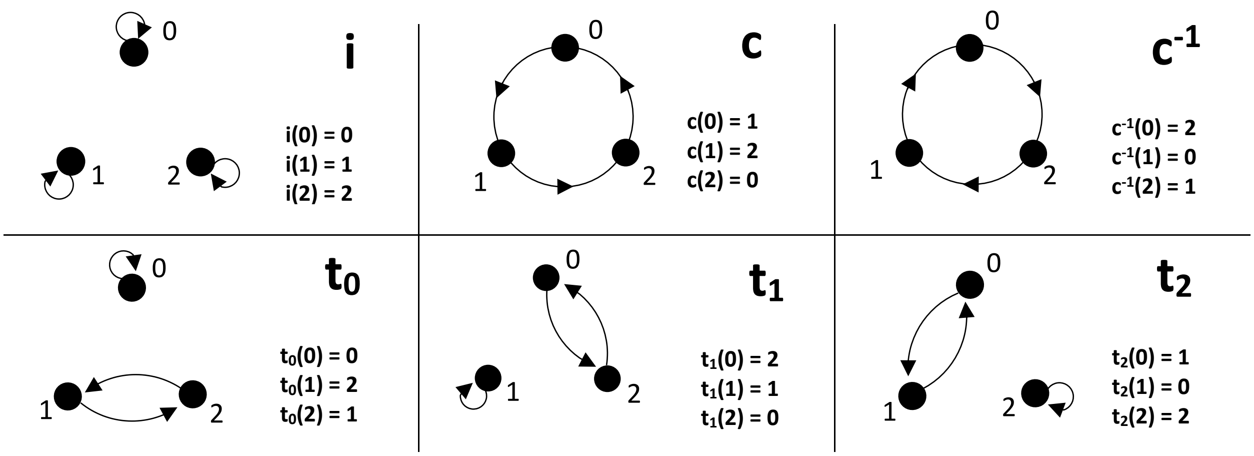


Fig 1: Six bijections of {0,1,2}, in function and graph notation

The correspondence of a particular permutation to a particular rearrangement is often established by the use of visual representation, in the form of illustrations of geometric figures being rotated or flipped in imaginary space. However, this correspondence may also be established notationally, first by accounting for transformations of elements with equations and then *inverting terms in their function notation.*

Example 1.1. (next page): The counter-clockwise rotation of a triangle in the plane is illustrated below, with numerals 0,1,2 labeling both the vertices of the figure and the positions those vertices occupy. The change in *vertices* resulting from the rotation is a cyclic permutation **f(012) = 201**, illustrated by superimposing two states of the triangle (before and after the transformation). The change in *positions* resulting from the rotation, on the other hand, is given by **f(012) = 120** (illustrated below by fixing the figure and superimposing two states of its positions). The two cycles expressing the rotation are transformable into one another by inverting numerals in their function notation.

[4] A purely differential element is asemantic – i.e., has no other structure.

[5] Noted as $\mathbf{t_0}$, $\mathbf{t_1}$, and $\mathbf{t_2}$, where $\mathbf{t_x}$ leaves element x fixed.

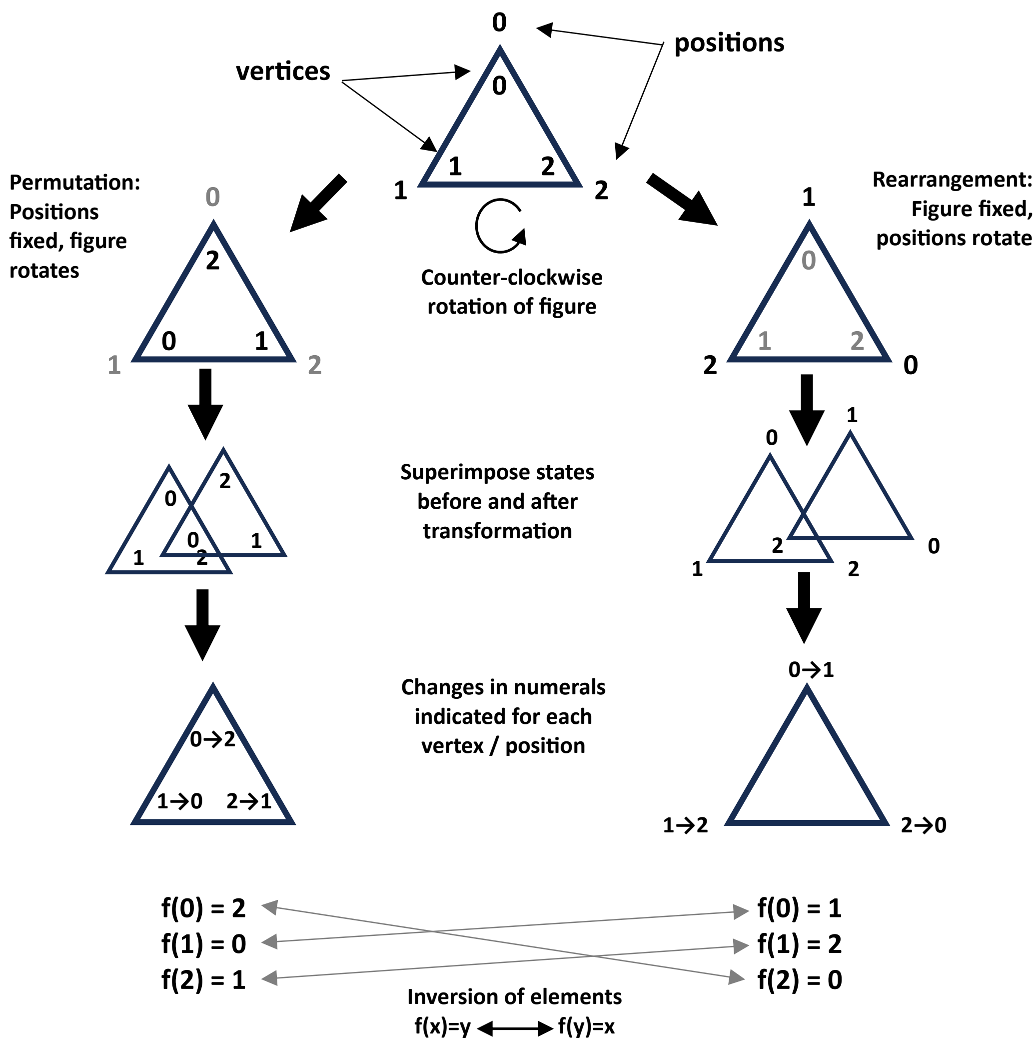


Fig. 2 – Visual aid model for permutation and rearrangement

We are motivated in this paper by the observation[6] that permutation and rearrangement can be distinguished by such notational inversions alone, without recourse to visual representation. As such, we define rearrangement as an inversion of elements in function notation denoting a bijection.

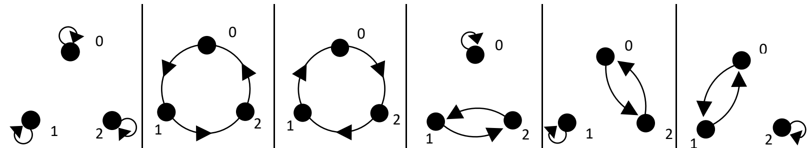
Definition 1.1. (Rearrangement): A *rearrangement* is an inversion $\varrho(\mathbf{f(a)=b}) \rightarrow \mathbf{(f(b)=a)}$ of a permutation **f(a)=b**, where **a,b** are elements of a set **A** and **f** is a function name for a bijection **f:A→A**. Rearrangements satisfy the symmetric relation $\varrho[\varrho(\mathbf{f(a)=b})] = (\mathbf{f(a)=b})$.

For now, we will use the term "permutation" to refer to a particular assignment of function names (denoted $\mathbf{P=\{i,c,c^{-1},t_0,t_1,t_2\}}$) to our set of bijections **Bij(X)** (ex. 1.0., above), and the term "rearrangement" for the set of inversions of this assignment according to def. 1.1.

[6] Hunter, p. 292.

Example 1.2. In Table 1 below, permutations of **X={0,1,2}** are given in the top two rows as graphs and stacked triplets of equations in function notation. Corresponding rearrangements are given in the bottom two rows by **ϱ(f(a)=b) → (f(b)=a)** for all **f ∈ Bij(X)** and **a**,**b ∈ X**. The middle row of the table assigns a single set of function names **{i,c,**c^{-1}**,**t_0**,**t_1**,**t_2**}** to both permutations and rearrangements of **X**, based on these inversions.

Permutations



f(a)=b	i(0)=0 i(1)=1 i(2)=2	c(0)=1 c(1)=2 c(2)=0	$c^{-1}(0)=2$ $c^{-1}(1)=0$ $c^{-1}(2)=1$	$t_0(0)=0$ $t_0(1)=2$ $t_0(2)=1$	$t_1(0)=2$ $t_1(1)=1$ $t_1(2)=0$	$t_2(0)=1$ $t_2(1)=0$ $t_2(2)=2$
	i	**c**	$\mathbf{c^{-1}}$	$\mathbf{t_0}$	$\mathbf{t_1}$	$\mathbf{t_2}$
ϱ(f(a)=b)	i(0)=0 i(1)=1 i(2)=2	c(1)=0 c(2)=1 c(0)=2	$c^{-1}(2)=0$ $c^{-1}(0)=1$ $c^{-1}(1)=2$	$t_0(0)=0$ $t_0(2)=1$ $t_0(1)=2$	$t_1(2)=0$ $t_1(1)=1$ $t_1(0)=2$	$t_2(1)=0$ $t_2(0)=1$ $t_2(2)=2$

Rearrangements

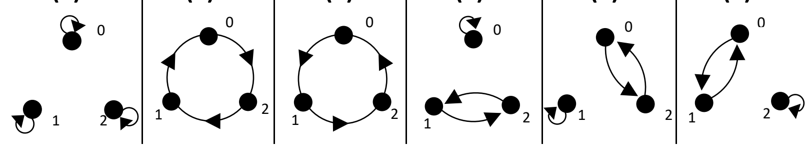


Table 1: Permutations and rearrangements of three elements

Table 1 indicates that permutation and rearrangement may be identified in a sort of *double reading* of a set of bijections. However, these two readings are indiscernible in terms of the transformed elements themselves (hence our arbitrary assignments above). In the next section, we will show how this double reading, which we have established for the synchronic structure of permutations as individual functions on a set, also appears for the diachronic structure of successive permutations in a group.

2. Group structure

A set of bijections of *n* elements can describe the totality of transformations of a *n*-sided regular polygon once the effect of successive mappings of that set is accounted for in a group structure, as follows:

Definition 2.0. A set **F** under a binary operation ***** and notated **(F*F)** or **(F,*)** is called an *algebraic group* if it meets the following requirements: 1) for all **f,g∈F**, their result **f*g** is an element of **F**; 2) there is an identity element **i** of **F** such that **i*f=f=f*i** for all **f∈F**; 3) every **f∈F** has an inverse element $\mathbf{f^{-1}}$ such that $\mathbf{f*f^{-1}=i=f^{-1}*f}$; and 4) for any **f,g,h∈F**, **(f*g)*h = f*(g*h)**.

Example 2.0. We bring our set of function names for **Bij(X)** (**P=**{**i,c,**c^{-1}**,**t_0**,**t_1**,**t_2}) under a binary operation ***** (called *composition*) such that **f*g= h** or **g(f)=h** means that permutation by **f**, then by **g**, is the same as permutation by **h** for any **f,g,h∈P**.[7] The resulting group **(P*P)** is denoted **(P,*)**.

All compositions of **(P,*)** are given below in Table 2. The table is read as follows: for any equation **g(f)=h**, the term inside the parentheses is chosen from the top row and the term to the left of the parentheses is chosen from the leftmost column. Their cross-referenced term in the body of the table is the result of the equation.

*	i	c	c^{-1}	t_0	t_1	t_2
i	i	c	c^{-1}	t_0	t_1	t_2
c	c	c^{-1}	i	t_2	t_0	t_1
c^{-1}	c^{-1}	i	c	t_1	t_2	t_0
t_0	t_0	t_1	t_2	i	c	c^{-1}
t_1	t_1	t_2	t_0	c^{-1}	i	c
t_2	t_2	t_0	t_1	c	c^{-1}	i

Example: $c(t_0)=t_2$

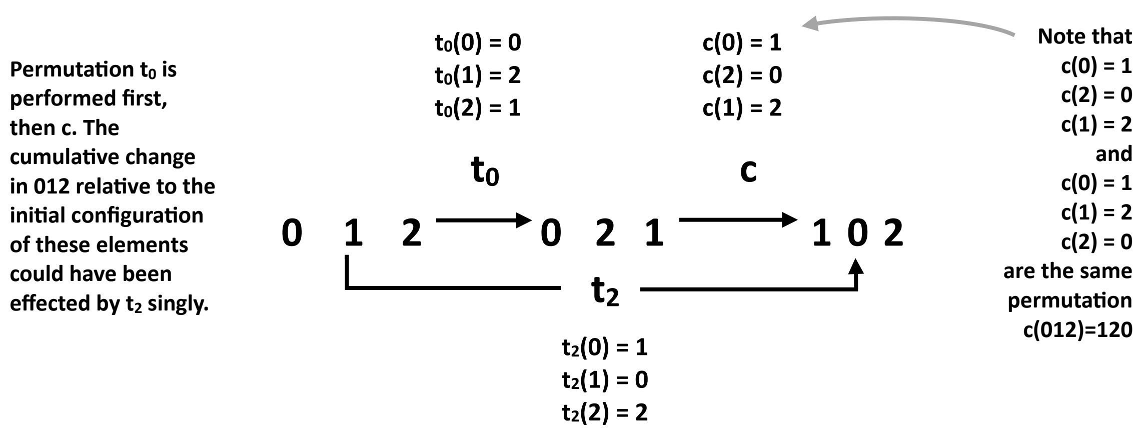
*	t_0
c	t_2

Table 2: Composition table for (P,*)

Table 2 shows that **(P,*)** is a group by def. 2.0. because **i** is the identity; **c** and $\mathbf{c^{-1}}$ are inverses to one another; and $\mathbf{t_0}$, $\mathbf{t_1}$, and $\mathbf{t_2}$ are their own inverses. For any three elements **f,g,h** of the group, **(f*g)*h = f*(g*h)**.

The inversion relating rearrangements to permutations of **Bij(X)**, which was based on the assignment of function names to bijections of **X** (Table 1, pg. 4), exists for **(P,*)** as an inversion of composed terms in equations. This inversion can be constructed in three ways: 1) by inverting the terms of an equation expressing a binary operation on **(P,*)**; 2) by "mirroring" the composition table for **(P,*)** along a diagonal axis; and 3) by mapping **(P,*)** to its opposite group.

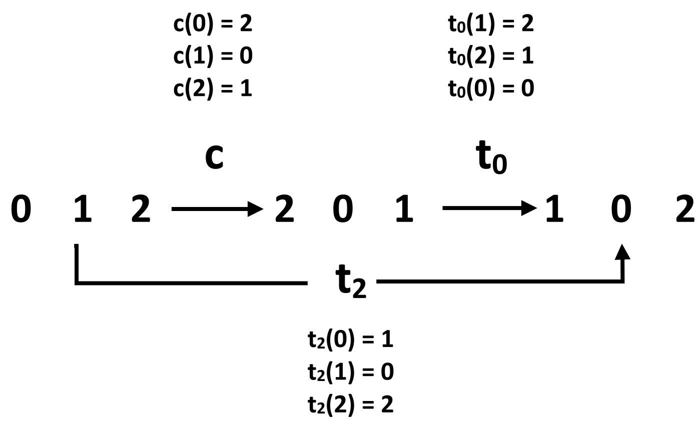
Example 2.1. Two successive permutations $\mathbf{t_0}$ and **c** are performed on an ordering of **X**, which has the same effect as the single permutation $\mathbf{t_2}$ (see Fig. 1, pg. 2):



[7] Our construction will notate composition by parentheses rather than concatenation with an asterisk unless the latter notation provides more clarity.

Following the assignments in Table 1 (pg. 4), the permutation $\mathbf{c(t_0)=t_2}$ above is inverted below into its corresponding rearrangement $\mathbf{t_0(c)=t_2}$, as follows:

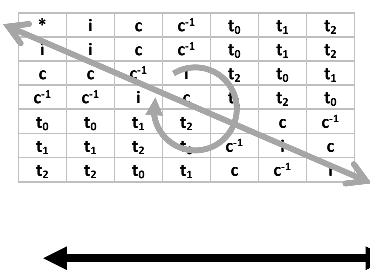
This inverted equation produces t_2 as its result due to the fact that c(012)=201 for rearrangements (see Table 1, pg. 4).



Example 2.2. Compositions of rearrangements of **X** are given below by mirroring the composition table for **(P,*)** along a diagonal axis going from top left to bottom right, then reading the table as done previously for permutations.

*	i	c	c^{-1}	t_0	t_1	t_2
i	i	c	c^{-1}	t_0	t_1	t_2
c	c	c^{-1}	i	t_2	t_0	t_1
c^{-1}	c^{-1}	i	c	t_1	t_2	t_0
t_0	t_0	t_1	t_2	i	c	c^{-1}
t_1	t_1	t_2	t_0	c^{-1}	i	c
t_2	t_2	t_0	t_1	c	c^{-1}	i

Composition table – Permutations



*	i	c	c^{-1}	t_0	t_1	t_2
i	i	c	c^{-1}	t_0	t_1	t_2
c	c	c^{-1}	i	t_1	t_2	t_0
c^{-1}	c^{-1}	i	c	t_2	t_0	t_1
t_0	t_0	t_2	t_1	i	c^{-1}	c
t_1	t_1	t_0	t_2	c	i	c^{-1}
t_2	t_2	t_1	t_0	c^{-1}	c	i

Composition table – Rearrangements

*	i	c	c^{-1}	t_0	t_1	t_2
i	i	c	c^{-1}	t_0	t_1	t_2
c	c	c^{-1}	i	t_1	t_2	t_0
c^{-1}	c^{-1}	i	c	t_2	t_0	t_1
t_0	t_0	t_2	t_1	i	c^{-1}	c
t_1	t_1	t_0	t_2	c	i	c^{-1}
t_2	t_2	t_1	t_0	c^{-1}	c	i

Example: $t_0(c)=t_2$

*	c
t_0	t_2

Table 3: "Mirrored" composition table for rearrangements of three elements

Mirroring the composition table for **(P,*)** essentially replaces each composition **a(b)** in the table with its inversion **b(a)**.[8] As such, we can define the structure of successive rearrangements of **X** as an *opposite group* to **(P,*)**, as follows:

Definition 2.1. An *opposite group* of a group **G=(G,*)** is a group $\mathbf{G^{op}} = \mathbf{(G,*')}$ that is isomorphic to **(G,*)** by the relation $\boldsymbol{\varrho : G \to G^{op}}$, or $\mathbf{(g_1 * g_2) = (g_2 *' g_1)}$ for all $\mathbf{g_1, g_2 \in G}$.

Example 2.3. We define the group of rearrangements of **X** as $\boldsymbol{\varrho : (P,*) \to (P,*)^{op}}$ such that **f*g = g*'f** for every composition of $\mathbf{f,g \in (P,*)}$. The binary operation for $\mathbf{(P,*)^{op}}$ will be the same as that of **(P,*)** (i.e., *** = *'**). Compositions in $\mathbf{(P,*)^{op}}$ can therefore be calculated by reading Table 3 (prev. page) in the same orientation as Table 2 (pg. 5).[9]

In constructing two algebraic groups for permutation and rearrangement, we have defined the relationship between these two transformations as a mapping between groups that preserves composition rules (i.e., an *isomorphism*[10] between **(P,*)** and $\mathbf{(P,*)^{op}}$), without recourse to visual models of objects already endowed with geometric properties from the beginning (triangles and other polyhedra, etc.).[11] By definition, this isomorphism is symmetric, such that we have once again arbitrarily assigned one group to permutations and the other to rearrangements. In consideration of the following outline given by Klein (1872) for the study of geometry:

> "Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group" (pg. 218),

we can identify certain requirements allowing us to construct a non-symmetric relation between these two groups – namely, an object ("manifoldness") and a group of transformations under which certain properties of an object remain unchanged.

In the next section, we construct a "simple geometry"[12] according to these conditions, where our group for permutations will function as an object and our group for rearrangements will function as a group of transformations for that object.

3. Construction

We begin by constructing a "manifoldness", or space, from the group we previously assigned to permutations of **X={0,1,2}** (namely, **(P,*)**), by dividing this group into pairs of functions based on each pair's relation to a fixed element $\mathbf{x \in X}$.

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[8] This inversion of *terms* is not the same as the inversion of *elements* described in def. 1.1. Notably, even though it might be predicted that the isomorphism between **(P,*)** and $\mathbf{(P,*)^{op}}$ would be $\mathbf{f \to f^{-1}}$ (considering that def. 1.1. defines rearrangement essentially as a renaming of a function by its group inverse), the transformation of one group's composition table into the other's is not accomplished by replacing each term in one table by its group inverse.

[9] Of course, these rearrangements can also be determined as "right compositions" in Table 2, where the function within parentheses is located instead in the left column and the term to the left of parentheses in the top row. Our choice to define our group of rearrangements as an opposite group simplifies the comparison (i.e., as two different "left compositions") of what are essentially two different readings of the same composition table.

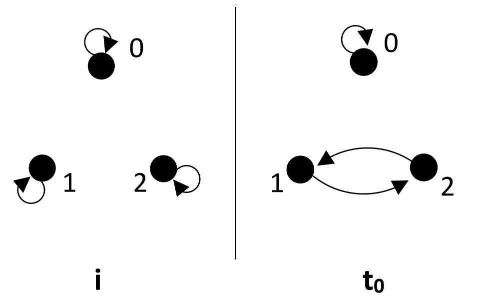
[10] An isomorphism is a reversible, structure-preserving map.

[11] Some examples of the use of such models may be taken from Budden, *Fascination of Groups*, pgs. 197ff; Grossman, I. & Magnus, W. *Groups and their Graphs*, pgs. 30ff; Carter, N. *Visual Group Theory*, pgs.74ff; for exceptions, see Armstrong, *Groups and Symmetry*, chapter 6 or Coxeter (1969). *Introduction to Geometry*, chapter 15.

[12] Vappereau, J.M. (1985). *Essaim: Le groupe fundamental du noeud*, Topologie en Extension, pg. 159.

Definition 3.0. (Fixed Element): Given our group **(P,*)**, an element $\mathbf{x \in X=\{0,1,2\}}$ is *fixed under permutation* for a subgroup[13] of functions $\mathbf{i,t_x \in (P,*)}$, where **i** is identity and $\mathbf{t_x}$ the transposition keeping **x** fixed.

Example 3.0. Element **0** of our set $\mathbf{X=\{0,1,2\}}$ will be chosen for the construction that follows. This element is fixed under two permutations forming the subgroup $\mathbf{H_0= \{i,t_0\}}$.



Definition 3.1. (Partition of a Group): Given a group **G** and a subgroup **H<G**, a *group partition* **G/H** is a set of cosets[14] $\mathbf{g*H = \{g*h: g \in G, h \in H\}}$ that is pairwise disjoint (i.e., no coset contains an element that is also in another coset of the same partition; and the union of cosets is all of the partition).

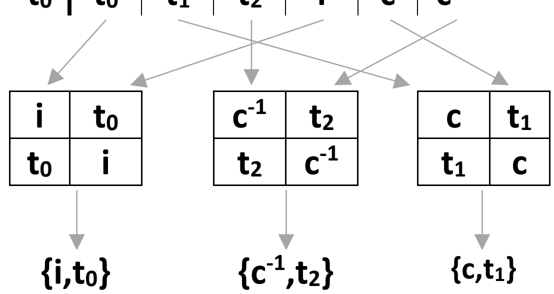
Example 3.1. Below, $\mathbf{H_0}$ partitions **(P,*)** by composing with each function $\mathbf{\{i,c,c^{-1},t_0,t_1,t_2\}}$ and generating a set of three pairwise disjoint cosets denoted $\mathbf{(P,*)/H_0}$.

(P,*) is partitioned by composition with H_0={i,t_0}, giving the set

(P,*)/H_0 = { {i,t_0}, {c^{-1},t_2}, {c,t_1} }

(see Table 2, pg.5)

*	i	c	c^{-1}	t_0	t_1	t_2
i	i	c	c^{-1}	t_0	t_1	t_2
t_0	t_0	t_1	t_2	i	c	c^{-1}



Cosets are unordered, such that, for example, {c^{-1},t_2} and {t_2,c^{-1}} are the same coset

Partitioning **(P,*)** by $\mathbf{H_0}$ endows the resulting cosets with the following property: each pair of functions in a coset maps the same element $\mathbf{x \in X}$ to **0** (see fig. 1, pg. 2).[15]

As such, $\mathbf{H_0}$ gives an *equivalence relation* on **(P,*)**, according to the following definition:

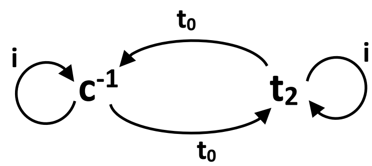
[13] A *subgroup* is a subset of functions of a group that also forms a group on its own under composition.
[14] A *coset* is a set of elements of a group that does not necessarily form a group under composition (i.e., a subgroup is a coset but not every coset is a subgroup).
[15] $\mathbf{(P,*)/H_0}$ is thus a set of *images* of a fixed element (**0**).

Definition 3.2. (Equivalence Relation): An *equivalence relation* **~** on a set **A** is a relationship of an element **a∈A** to itself or to other elements **b,c∈A** that is *reflexive*, *symmetric*, and *transitive*, by the following formulas:

3.2.1. (reflexive): **a~a** for all **a∈A**;
3.2.2. (symmetric): **a~b** implies **b~a** for all **a,b∈A**; and
3.2.3. (transitive): **a~b** and **b~c** implies **a~c** for all **a,b,c∈A**.

The set of all elements defined by an equivalence relation is called an *equivalence class*.

Example 3.2. The relations diagramed below for coset $\{c^{-1},t_2\}$ are generalizable to the other cosets of $(P,*)/H_0$, where reflexivity is given by **i**, symmetry by t_0, and transitivity by both functions (e.g., $t_0(i(c^{-1}))=t_0(t_2)$).



*	c^{-1}	t_2
i	c^{-1}	t_2
t_0	t_2	c^{-1}

By partitioning our group for permutations, we have created an *object* out of **(P,*)**, in the sense that the group has been divided into *equivalence classes of images* of a fixed element of our original set of elements **X={0,1,2}**. The identity of each coset of the partition is given by composition with H_0, as shown by the above graph. As such, the identity relation given to each coset by this subgroup is intrinsic to the partition, as it does not distinguish one equivalence class from another (e.g., any of the three pairs of functions could be substituted for c^{-1} and t_2 in the diagram above).

Taking this object $(P,*)/H_0$ as a *set of points* for our construction,[16] we can distinguish each equivalence class of our partition by giving this set of points an *extrinsic* structure – namely, its transformation under our group for rearrangements $(P,*)^{op}$, according to the following definition of a *group action*:

Definition 3.3. A group action **∝ : (G × A)→A** is a map of a set **A** to itself by the operation of a group **G**. Group actions meet the following requirements:

3.3.1. An identity element **e** such that **e × a = a** for all **a ∈ A**. This is called a *trivial action*.
3.3.2. The product rule of **G** is preserved in the action, such that two functions **f,g ∈ G** acting successively on an element **a ∈ A** will have the same effect as their result[17] **f*g** acting on that same element (i.e., **g × (f × a) = f*g × a** for all **f,g ∈ G** and **a ∈ A**).

[16] "We always regard the totality of configurations in space as simultaneously affected by the transformations, and speak therefore of *transformations of space*. The transformations may introduce other elements in place of points" (Klein, 1893, pg. 218). Also: "The basic spatial elements of a geometry are therefore not genuine geometrical objects such as points or lines, but rather tuples of numbers assigned to the variables in question" and "Transformations of a space are thus represented as transformations of coordinates […] specified in terms of […] algebraic equations describing the functions between the coordinates" (Schiemer, 2020, pgs. 119-120).
[17] Note that, for an equation **f*g=h**, the term **f*g** may be used as a substitutive notation for its result **h**.

Notationally, a group action may be constructed in a table, similar to a group composition table, with the elements of the group in the left column and elements of the set along the top row. The results of the action are then cross-referenced in the body of the table.

$(G \times A) \rightarrow A$:

$\times$	A
G	A

Below, we construct a group action on **$(P,*)/H_0$**, and show that this action is valid by def. 3.3. under **$(P,*)^{op}$**, our group for rearrangements.

Example 3.3. (Table construction): Table 4 below describes the group action **$\varphi : ((P,*)^{op} \times (P,*)/H_0) \rightarrow (P,*)/H_0$**, where **$(g \times \{x,y\}) \rightarrow \{x',y'\}$** is an action determined by the composition of a function **$g \in (P,*)^{op}$** (left column) with functions **x , y** of a coset **$\{x,y\} \in (P,*)/H_0$** (top row) giving a coset **$\{x',y'\} \in (P,*)/H_0$** (body of table).[18]

Every action of φ on unordered cosets of $(P,*)/H_0$ is determined by ordered compositions in $(P,*)^{op}$

Example: $(t_1 \times \{i,t_0\}) \rightarrow \{c,t_1\}$:

*	i	t_0
t_1	t_1	c

$\times$	$\{i,t_0\}$	$\{c^{-1},t_2\}$	$\{c,t_1\}$
i	$\{i,t_0\}$	$\{c^{-1},t_2\}$	$\{c,t_1\}$
c	$\{c,t_1\}$	$\{i,t_0\}$	$\{c^{-1},t_2\}$
c^{-1}	$\{c^{-1},t_2\}$	$\{c,t_1\}$	$\{i,t_0\}$
t_0	$\{i,t_0\}$	$\{c,t_1\}$	$\{c^{-1},t_2\}$
t_1	$\{c,t_1\}$	$\{c^{-1},t_2\}$	$\{i,t_0\}$
t_2	$\{c^{-1},t_2\}$	$\{i,t_0\}$	$\{c,t_1\}$

Table 4: Group action $\varphi : ((P,*)^{op} \times (P,*)/H_0) \rightarrow (P,*)/H_0$

We can see that φ is a valid self-map by the action of our group for rearrangements **$(P,*)^{op}$** (i.e., the result of any action on a coset of **$(P,*)/H_0$** is a coset of **$(P,*)/H_0$**).[19] Additionally, each coset has two trivial actions (identity and a transposition). Finally, actions of φ preserve the product rule for **$(P,*)^{op}$** (e.g., **$c \times (t_0 \times \{c,t_1\}) = (c*t_0) \times \{c,t_1\} = t_1 \times \{c,t_1\} = \{i,t_0\}$**).

Below, we give an action diagram for φ composed of nodes encircling each coset of **$(P,*)/H_0$** and arrows describing each transformation under φ.

[18] The composition table for **$(P,*)^{op}$** is produced here for reference:

*	i	c	c^{-1}	t_0	t_1	t_2
i	i	c	c^{-1}	t_0	t_1	t_2
c	c	c^{-1}	i	t_1	t_2	t_0
c^{-1}	c^{-1}	i	c	t_2	t_0	t_1
t_0	t_0	t_2	t_1	i	c^{-1}	c
t_1	t_1	t_0	t_2	c	i	c^{-1}
t_2	t_2	t_1	t_0	c^{-1}	c	i

[19] It is left to the reader to calculate, based on the construction that follows, that the action **$\varphi : ((P,*) \times (P,*)/H_0) \rightarrow (P,*)/H_0$**, for example, is *not* a valid self-map (i.e., it produces cosets that are not elements of **$(P,*)/H_0$**).

Example 3.5. (Action Diagram): The group action **φ : ((P,*)^{op} × (P,*)/H$_0$) → (P,*)/H$_0$** is transcribed below in a graph where each coset of **(P,*)/H$_0$** is a node and each action of **φ** is an arrow. The graph describes a valid group action by def. 3.3. because it is complete (i.e., gives a self-map **(P,*)/H$_0$ → (P,*)/H$_0$**; gives a looped arrow for each node (3.3.1.); and composes arrows (3.3.2.).[20]

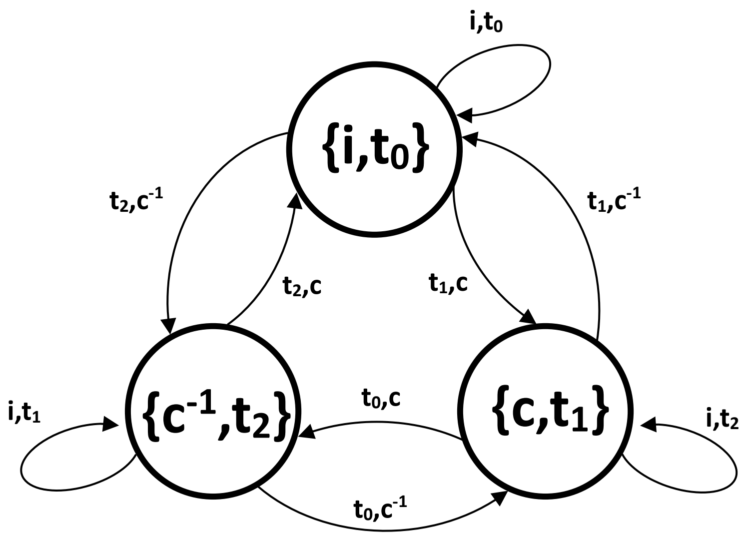


Fig. 3 – Action diagram for φ

Under **φ**, the space composed of our three points is *homogenous* (i.e., any point may be reached from any other point in the system) since **φ** acts *transitively* (i.e., for any two points **P$_1$**, **P$_2$**, there exists a transformation **f** such that **(f × P$_1$) → P$_2$**). As such, the basic structure of our object – namely, the three distinct points of our space – is unchanged by the action of **φ**.

Recalling the conditions noted above (pg. 7) for defining a geometry (an object, a group of transformations, and invariant properties of the object under transformation) we give the following definition of a geometry of permutation and rearrangement:

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[20] For example, the composition **(t_0,c^{-1}) * (t_2,c^{-1}) = (t_1,c)** is an equality of two-arrowed and one-arrowed paths connecting nodes **{i,t_0}** and **{c,t_1}** (composing arrows also shows preservation of the product rule for **(P,*)^{op}**). The composition table for **(P,*)^{op}** (pg.6) is again reproduced for reference:

*	i	c	c^{-1}	t_0	t_1	t_2
i	i	c	c^{-1}	t_0	t_1	t_2
c	c	c^{-1}	i	t_1	t_2	t_0
c^{-1}	c^{-1}	i	c	t_2	t_0	t_1
t_0	t_0	t_2	t_1	i	c^{-1}	c
t_1	t_1	t_0	t_2	c	i	c^{-1}
t_2	t_2	t_1	t_0	c^{-1}	c	i

Definition 3.4. Given a group **G**, a subgroup **H≤G**, and a group partition **G/H**, we define a *geometry of **G/H*** as a G-set[21] **(G/H , α)** where **α** is a group action on the cosets of **G/H** meeting the following requirements:

3.4.1. (distinct points): each coset of **G/H** is mapped to itself by a unique stabilizing action **(g × {x}) → {x}**, where **g ∈ G** and **{x} ∈ G/H**.

3.4.2. (transitive action): for any two cosets **{x}, {y} ∈ G/H**, there exists a function **g ∈ G** such that **(g × {x}) → {y}**.

Invariance of this "three distinct points" property of **(P,*)/H_0** under **φ** implies that transformations of cosets of **(P,*)/H_0** can be *represented as permutations* of undifferentiated (i.e., asemantic) elements. Such a representation[22] is based on the following theorem concerning finite algebraic groups generally:

Theorem 3.0. (Cayley's Theorem): Every finite group is isomorphic to (i.e., shares structure with) a subgroup of a symmetric group (i.e., the group of all bijections of a set of elements).

A proof of this theorem is presented in Appendix A. A visual reference for the representation of our geometry **(G/H , α)** by a group of bijections is given below, where Table 4 (pg. 10) is reproduced on the left, with a counterpart table on the right derived from Table 4 by substituting **{0,1,2}** for cosets of **(P,*)/H_0**. This substituted table gives orderings previously assigned to rearrangements of **X** (see Table 1, pg. 4).

×	$\{i,t_0\}$	$\{c^{-1},t_2\}$	$\{c,t_1\}$
i	$\{i,t_0\}$	$\{c^{-1},t_2\}$	$\{c,t_1\}$
c	$\{c,t_1\}$	$\{i,t_0\}$	$\{c^{-1},t_2\}$
c^{-1}	$\{c^{-1},t_2\}$	$\{c,t_1\}$	$\{i,t_0\}$
t_0	$\{i,t_0\}$	$\{c,t_1\}$	$\{c^{-1},t_2\}$
t_1	$\{c,t_1\}$	$\{c^{-1},t_2\}$	$\{i,t_0\}$
t_2	$\{c^{-1},t_2\}$	$\{i,t_0\}$	$\{c,t_1\}$

$\frac{0}{\{i,t_0\}}$ $\frac{1}{\{c^{-1},t_2\}}$ $\frac{2}{\{c,t_1\}}$ ⟷

	0	1	2
i	0	1	2
c	2	0	1
c^{-1}	1	2	0
t_0	0	2	1
t_1	2	1	0
t_2	1	0	2

As such, we discover the structure of our original bijections (i.e., self-mapping of a set of elements) prefigured in the structure of the group action (i.e., transitive self-mapping of a set of points) determining our geometry (**(P,*)/H_0**, φ).

This isomorphism between **φ** and **Bij(X)** implies that (**(P,*)/H_0**, φ) is a geometry of *rigid motions*, meaning that an object will be "the same" under its group of space-transformations if it has an

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[21] A G-set is a pair **(A,α)** where **A** is a set and **α** is an action on **A** by a group **G** such that **g × (f×a) = fg×a** for all **f,g ∈ G** and **a ∈ A** (see def. 3.3.2, above).

[22] Specifically, a *permutation representation*, defined as a *homomorphism* (structure-preserving map) **δ:G→S_n**, where **G** is a group and **S_n** the group of all bijections of a set of **n** elements (otherwise known as a *symmetric group*). Compositions of **G** are preserved by this mapping such that **δ(x) ○ δ(y) = δ(xy)** for all **x,y** of the set (○ denotes the group operation for **S_n** and concatenation (e.g., **xy**) denotes the group operation for **G**). Our group action **φ** represents **$(P,*)^{op}$**, which is essentially the symmetric group for *rearrangements* of **X**.

equivalence class of images by properties remaining invariant under symmetric transformations (reflections and rotations).[23]

One final implication of the shared structure of φ and **Bij(X)** is a sort of *double reading* of our geometry, insofar as it describes the rigid motions of a 3-sided regular polygon in two ways. In Fig. 4 below, Fig. 3 is presented as a sort of *composite graph* of rearrangements of 3 elements, with a single triangle model superimposed on its three points. In Fig. 5, the same graph is presented as a *composite algorithm* for the entire group of rigid transformations on all 6 permuted states of a triangle model.

Example 3.6. (Composite graph reading): Below, Fig. 3 (pg. 12) is presented as a composite of all 6 rearrangement graphs (Table 1, pg. 4), with an example transformation (**t_2**) given on the right, where **t_2** acts on points **P_0** and **P_1**, flipping a triangle model superimposed on the graph.

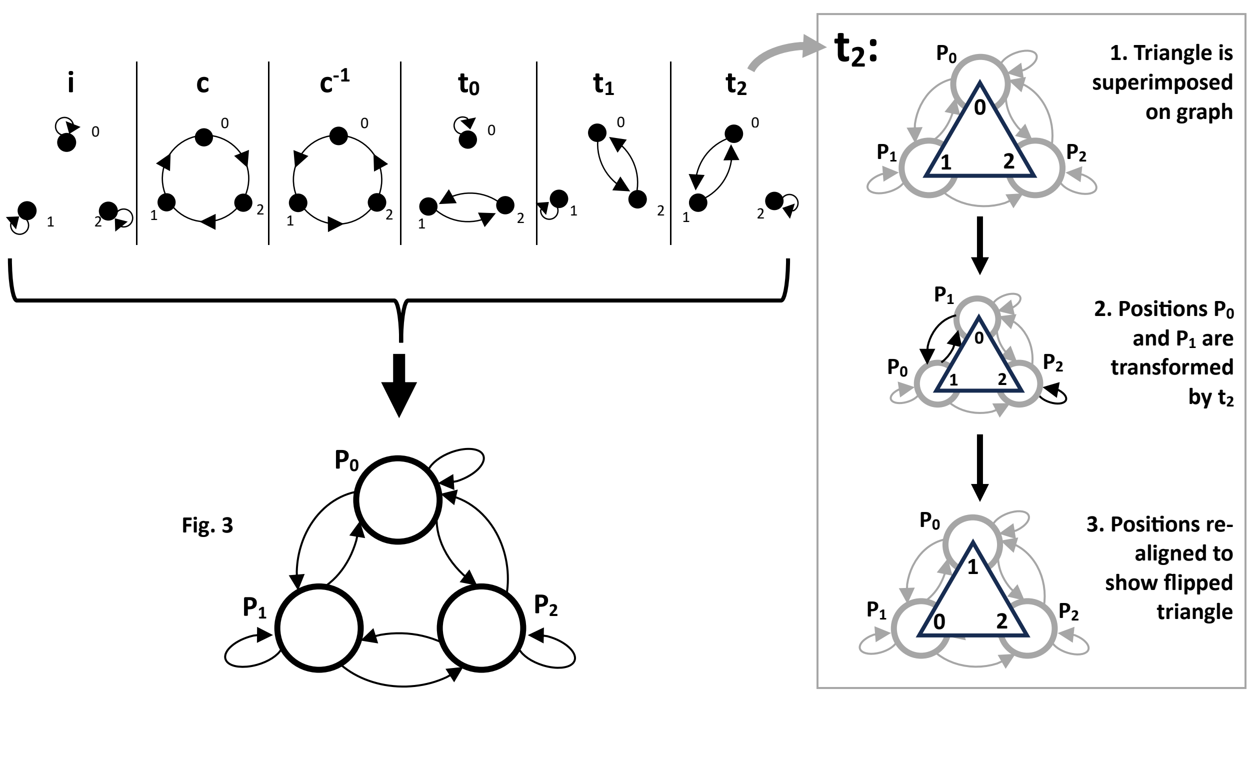

Fig. 4 – "composite graph" reading of Fig. 3 (pg. 12)

Example 3.7. (Composite algorithm reading): Fig. 3 (pg. 11) is reproduced on the next page with the following modifications: a pair of triangle models (Fig. 2, pg. 3) is substituted into each node of the graph for each coset of **(P,*)/H**, using the assignments in Table 1 (pg. 4) of functions

[23] An example of a non-rigid geometry is given by *projective* transformations on a set of points, where, for example, two images of a triangle that are not symmetrical to each other (e.g., skewed images) would still be part of the same equivalence class determining the identity of that triangle.

($\{\mathbf{i,c,c^{-1},t_0,t_1,t_2}\}$) to *permutations* of **{0,1,2}**. The arrows of the graph act on the positions of each triangle model, *rearranging* $\mathbf{P_0,P_1,P_2}$. according to assignments of functions in Table 1 (pg.4). In the breakout box (below, right), this action is described in detail for arrow $\mathbf{c^{-1}}$ on triangle **c**.[24]

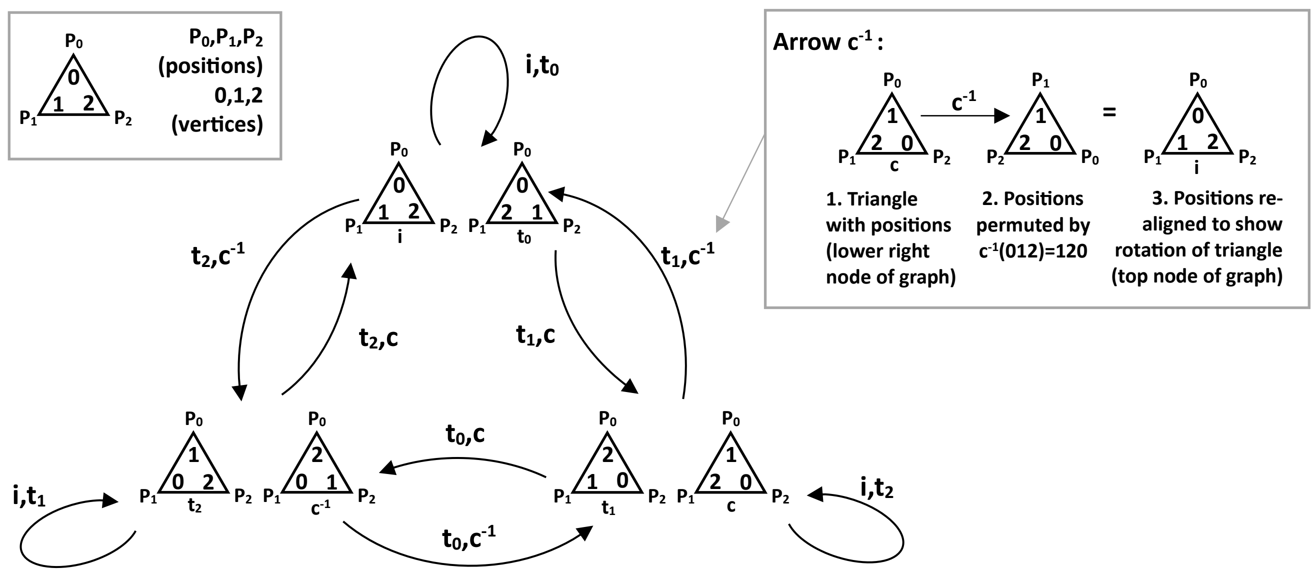

Fig. 5 – "composite algorithm" reading of Fig. 3 (pg. 12)

In this double reading,[25] where we can see an "oscillation between representations of things and a representation of classes of transformations between these things", we have attempted to give an example of "the distinction introduced by Freud between representations of things and representations of words"[26] in a construction that simultaneously depicts and describes the rigid geometry of a triangle.

[24] In confirming calculations for the other actions on the graph, the reader is reminded that the arrows of the graph follow the table we have chosen for *rearrangements* of **X={0,1,2}** (pg. 7), where $\mathbf{c^{-1}(012)=120}$ (see pg. 4). This distinction explains a difference between our presentation in Fig. 3 and that of Vappereau (1985, pg. 158), who preserves the "permutation" sense of the cycles of the group (i.e., **c(012)=120**).

[25] In which one reading (Fig. 4) responds to the synchronic indiscernibility noted in Sect. 1 between permutation and rearrangement, and another (Fig. 5) to the diachronic indiscernibles ($\mathbf{(P,*)}$, $\mathbf{(P,*)^{op}}$) identified in Sect. 2.

[26] Vappereau, *Essaim*, pg. 160.

Appendix B – Cayley's Theorem (proof)

TO PROVE: Every finite algebraic group is isomorphic to a subgroup of a symmetry group.
To prove this theorem, we must show that 1) elements of a group G of order *n* can be mapped to a subgroup of a symmetry group of *n* elements, notated S_n.

PART 1: Left composition of G with itself is bijective. We begin with a notation for left composition:

1a):	Let $\lambda_f(g) = fg$ for all $f,g \in G$	$\lambda_f(g)$ = left composition of f with g

CONDITIONS: $\lambda_g(f)$ is a bijection iff it is 1) injective; and 2) surjective.

1.1. $\lambda_f(g)$ is injective iff the result of the left composition fg is unique – that is, for any $f,g_1,g_2 \in G$, $\lambda_f(g_1) = \lambda_f(g_2)$ implies that $g_1=g_2$.

1b):	Given $f,g_1,g_2 \in G$	Given
1c):	$\lambda_f(g_1) = \lambda_f(g_2) \Longrightarrow fg_1 = fg_2$	By 1a, above
1d):	$fg_1 = fg_2 \Longrightarrow g_1 = g_2$	By left cancelation
1e):	$\lambda_f(g_1) = \lambda_f(g_2) \Longrightarrow g_1 = g_2$	Implication: 1c – 1d (injective property)

1.2. $\lambda_f(g)$ is surjective iff, for any $h \in G$, there exists an $f,g \in G$ such that $\lambda_f(g) = h$

1f):	for $f,g,h \in G$, let $\lambda_f(g) = fg$	Given / 1a, above
1g):	let $g = f^{-1}h$	Given
1h):	then $fg = f(f^{-1}h)$	Substitution on 1g by 1h
1i):	$f(f^{-1}h) = (ff^{-1})h$	Associative property of G
1j):	$(ff^{-1})h = h$	Inverse / identity properties of G
1k):	$\lambda_f(g) = h$	Implication: 1i=1k (surjective property)

By (1.1) and (1.2.), left composition on G is a bijection (i.e., $\lambda : G \rightarrow G$).

PART 2: Since λ is bijective, we want to show that G can be represented by a group structure on bijections of G. We can show this by proving that the product rule for G is preserved for a group of bijections $Bij_{|G|}$. In other words, we must prove there is a homomorphism $\Phi : G \rightarrow Bij_{|G|}$ by mappings $\Phi(f) = \lambda_f$, where $f \in G$ and $\lambda_f \in Bij_{|G|}$; specifically, we must prove that $\lambda_{fg} = \lambda_f \circ \lambda_g$, where $\circ$ denotes composition of bijections and fg (i.e., concatenation of letters) denotes composition in G.

TO PROVE: $\lambda_{fg} = \lambda_f \circ \lambda_g$

2a):	Given $f,g,x \in G$	Given
2b):	$\Phi(fg) = \lambda_{fg}$	Mapping of $fg \in G$ to $fg \in Bij_{|G|}$
2c):	$\lambda_{fg}(x) = fg(x)$	By 1a, above
2d):	$fg(x) = f(gx)$	Associativity axiom of G
2e):	$f(gx) = \lambda_f(gx)$	1a
2f):	$\lambda_f(gx) = \lambda_f(\lambda_g(x))$	1a
2g):	$\lambda_f(\lambda_g(x)) = (\lambda_f \circ \lambda_g)x$	Associativity axiom of $Bij_{|G|}$
2h):	$\lambda_{fg}(x) = (\lambda_f \circ \lambda_g)x$	Implication: 2c – 2g
2i):	$\lambda_{fg} = \lambda_f \circ \lambda_g$	Right cancelation: Homomorphism